WEST BENGAL STATE UNIVERSITY

B.Sc. Honours/Programme 3rd Semester Examination, 2021-22

MTMHGEC03T/MTMGCOR03T-MATHEMATICS (GE3/DSC3)

REAL ANALYSIS

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Find the least upper bound of the set $S=\left\{\frac{1}{p}+\frac{1}{q}\ :\ p,\, q\in\mathbb{N}\right\}$.

(b) Prove that $\mathbb{N}$ is not bounded above.

(c) Show that 0 is a cluster point of the set $S=\left\{\frac{1}{n}\ :\ n\in\mathbb{N}\right\}$.

(d) Show that $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$.

(e) Examine the convergence of the series $\sum\limits_{n=1}^{\infty}\dfrac{(n\,!)^2}{(2n)!}$.

(f) Examine whether the sequence of functions $\{f_n\}$ converges uniformly on $\mathbb{R}$, where for all $n\in\mathbb{N}$,

$$f_n(x)=\frac{x}{n}\ ,\ \text{for all } x\in\mathbb{R}.$$

(g) Show that the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^3+n^4x^2}$ is uniformly convergent on $\mathbb{R}$.

(h) Find the radius of convergence of the power series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n n!}{n^n}x^n$.

(i) Show that the sequence $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence.

2. (a) If S is a non empty subset of $\mathbb{R}$ and also bounded below then prove that S has an infimum. 3

(b) Show that the subset $S=\{x\in\mathbb{Q}\ :\ x>0\,,\ x^2<2\}$ is a non empty subset of $\mathbb{Q}$, bounded below; but $\inf S$ does not belong to $\mathbb{Q}$. 5

3. (a) Show that 0 is a limit point of the set $\{x\ :\ 0<x<1\}$. 2

(b) Find all limit points of the set of all rational numbers $\mathbb{Q}$. 3

(c) Prove that $\mathbb{Z}$ is not bounded below. 3

4. (a) Prove that the set of all open intervals having rational end points is enumerable. 4

(b) Show that the sequence $\left\{\dfrac{n^2+2022}{n^2}\right\}$ converges to 1. 4

5. (a) Show that the sequence $\{x_n\}$ is monotone increasing, where 4

$$x_n = 1 + \tfrac{1}{2} + \ldots\ldots + \tfrac{1}{n} \quad \text{for all } n \in \mathbb{N}$$

Hence show that the sequence $\{x_n\}$ is not convergent.

(b) Apply Cauchy's criterion for convergence to show that the sequence $\{x_n\}$ is convergent, where 4

$$x_n = 1 + \tfrac{1}{2!} + \ldots\ldots + \tfrac{1}{n!} \quad \forall\ n \in \mathbb{N}$$

6. (a) Let $x \in \mathbb{R}$. Show that the series $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$ converges absolutely. 4

(b) Examine the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}$. 4

7. (a) Discuss the convergence of the series $\sum 1/n^p$, $p > 0$. 4

(b) Let $f_n(x) = x^n$, $x \in [0, 1]$. Show that the sequence of function $\{f_n\}$ is not uniformly convergent on [0, 1]. 4

8. (a) Let $f_n(x) = nxe^{-nx^2}$, $x \in [0, 1]$, $n \in \mathbb{N}$. Show that the sequence $\{f_n\}$ is not uniformly convergent on [0, 1]. 4

(b) Prove that the series $\sum \frac{x}{n + n^2x^2}$ is uniformly convergent for all real x. 4

9. (a) Show that the series $x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \ldots\ldots$ is not uniformly convergent on [0, 1]. 4

(b) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly continuous on $\mathbb{R}$. For each $n \in \mathbb{N}$, let $f_n(x) = f\left(x + \frac{1}{n}\right)$, $x \in \mathbb{R}$. Prove that the sequence $\{f_n\}$ is uniformly convergent on $\mathbb{R}$. 4

10.(a) If $\{u_n\}$ be a sequence of real numbers and $\sum u_n^2$ is convergent prove that $\sum \frac{u_n}{n}$ is absolutely convergent. 4

(b) If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences then prove that, 4

(i) $\{x_n + y_n\}$ is a Cauchy sequence

(ii) $\{x_n y_n\}$ is a Cauchy sequence.

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

——×——